International Journal of Business and Statistical Analysis
ISSN (2384-4663)
Int. J. Bus. Stat. Ana. 2, No. 1 (Jan-2015)

Nonparametric Estimation for Density Quantile Function Via TL-Moments

E. M. Abd-Elrazek[1], A.M. Qandeel[1] and S.A. Habib[1]

Department of Statistics, Faculty of Commerce, Benha University

Received 6 December 2014, Revised 17 December 2014, Accepted 18 December 2014, 1 January 2015

Abstract: The estimation of population density quantile is of great interest when a parametric form for the underlying distribution is not available. In addition, the density quantile function could be used as an alternative of density function. This paper provides a practical description of new nonparametric estimator of density quantile function based on TL-moments methods and its asymptotic properties are studied.

Keywords: quantiles, density ,quantile density function,TL-moments, kernel density estimators.

1. Introduction

Density estimation has experienced a wide explosion of interest over the last 20 years. Silverman's (1986) book on this topic has been cited over 2000 times. In classical statistics, most of distributions are defined in terms of their cumulative distribution function (cdf) or probability density function (pdf). There are some distributions which do not have the cdf or pdf in an explicit form but a closed form of the quantile function is available, for example Generalized Lambda distribution and Skew logistic distribution (Gilchrist (2000) and Karian and Dudewicz (2000). In addition, quantile measures are less influenced by extreme observations. Hence the quantile function can also be looked upon as an alternative to the distribution function in data for heavy tailed distributions. The quantile function approach is a useful tool in statistical analysis.

For measuring descriptive features of a univariate distribution, the method of moments is very popular, but their use is confined to sufficiently light-tailed distributions; see, Bera and Bilias (2002). An appealing alternative is provided by the series of L-moments and TL-moments, which have the form of expectations of selected linear functions of order statistics. The L-moments and TL-moments have an attractive properties not shared by the method of moments; see, Hosking (1990). For example, L-moments of any order k exists under merely a finite first moment assumption, making the entire series of L-moments available for heavy-tailed distributions. TL-moments is defined under weaker assumption of L-moments where it does not require a finite first moment assumption. Further, the L-moments and TL-moments completely determine the parent distribution; see, Elamir and Seheult (2004).

As interest in statistical modeling using heavy-tailed distributions is increasing, so is the important of the potential by the L-moment and TL-moment approaches. The need of linear moments has been developed in support of regional frequency analysis in environmental science, which treats the quantile of distributions of variables such as annual maximum precipitation, solid flow, wind speed observed at each site in a given network. Hosing and Wallis (1997) provide an excellent exposition. Also linear moments approach has special utility in applications where descriptive estimates (location, spread, skewness, and kurtosis) more stable than the usual central moments are critically needed. Such concern arise, for example, in volatility estimation in financial risk management involving market variables such as stock indices, interest rates; see, Serfling (1980) , Embrechts et al. (1997), Leonowicz et al. (2005) and Willinger et al. (1998).

Hosking (2007) has derived of approximation to the quantile function in terms of population TL-moments. Elamir (2010) introduced properties of this approximate

Email: enayat15@yahoo.com, abdelfattah.kandil@gmail.com, darali10@gmail.com

function by minimizing the weighted mean square error between the population quantile function and its TL-moments representation. Also, he studied properties of the corresponding sample estimator. He concluded that the estimators have a good approximation to population quantile for a broad class of probability distribution functions. Also, Elamir (2010) derived an optimal choice for the amount of trimming from known distributions, based on the minimum sum of the absolute value of the errors between the quantile probability function and its TL-moments representation. We use and apply TL-moments to obtain the density quantile function and estimate it.

In section 2, we defined density quantile function. In section 3 we review the TL-moments and sample TL-moments. In section 4 important transformations are given. Representation of density quantile function in terms of TL-moments is given in section 5. Estimation of density quantile function from sample is given in section 6. Finally, the applications of the estimated density quantile to some known distributions and Breast Tumors data are given in Section 7.

2. Density quantile function

Given a random variable X representing some one-dimensional population quantity, the population quantile function Q is defined as $Q_X(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in [0,1]$, where $F = F(x)$ is the population cdf. If F is absolutely continuous with density $f = f(x)$ and is one-to-one, Q is differentiable on the open unit interval, $(0,1)$, and $q(u) = Q'(u)$, $u \in [0,1]$, is called the quantile density function (qdf). In this case $F(Q(u)) = u$. Differentiation on both sides gives $q(u)\, f(Q(u)) \equiv 1$. The function

$$fQ(u) = f(Q(u))$$

is called the density quantile function; see Parzen (1979), Cheng and Parzen (1997) and Cheng (2002). Thus, the density quantile function is another related quantity which obtained from the pdf, $f(x)$, by substituting for x with the quantile function, as

$$f_u(u) = f(Q(u)) \qquad (1),$$

see Parzen (1979).

As $x = Q(u)$ and $u = F(x)$ for any pair of values (x,u), it follows from the definition of differentiation that

$$\frac{dx/du}{du/dx} = 1 \text{ , so } \qquad \frac{dQ(u)/du}{dF(x)/dx} = 1,$$

hence $q(u)\, f(x) = 1$ and, therefore, expressing all in terms of u, $q(u)\, f_u(u) = 1$. The two functions $q(u)$ and $f_u(u)$ are thus reciprocals of each other. Thus, the density-quantile function can be defined in terms of quantile density function as

$$f(Q(u)) = \frac{1}{q(u)} \qquad (2).$$

Otherwise, the quantile density function can be defined in terms of density quantile function as

$$q(u) = \frac{1}{f(Q(u))} \text{ ……..} \qquad (3),$$

see Parzen (1979) and Jones (1992).

3. Trimmed L-moments (TL-moments)

Let X be a real-valued random variable with distribution function $F(x) = F_X = F$ and quantile function $Q(u) = F^{-1}(u) = x(F)$, and let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of X . Thus, the rth TL-moment $\lambda_r^{(t_1,t_2)}$ in terms of expected value is given as

$$\lambda_r^{(t_1,t_2)} = r^{-1} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \mathrm{E}(X_{r+t_1-j:r+t_1+t_2}) \ ,$$

$$r = 1,2,\ldots,\quad t_1, t_2 = 0,1,2,\ldots \qquad (4),$$

where t_1 and t_2 are the amounts of lower and upper trimming and The expectation of $X_{j:r}$ can be written as

$$\mathrm{E}(X_{j:r}) = j\binom{r}{j}\int_0^1 Q(u) u^{j-1}(1-u)^{r-j}\, du \ , \ j \leq r \qquad (5)$$

(see David (2003)).

Another form of TL-moments is using the standard form of expected value $E(X_{j,r})$ are

$$\lambda_r^{(t_1,t_2)} = r^{-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\frac{(r+t_1+t_2)!}{(r+t_1-j-1)!(t_2+j)!}\int_0^1 Q(u) u^{r+t_1-j-1}(1-u)^{t_2+j}\,du$$

$$= r^{-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\frac{(r+t_1+t_2)!}{(r+t_1-j-1)!(t_2+j)!}E\left[Q(u) u^{r+t_1-j-1}(1-u)^{t_2+j}\right] \qquad (6)$$

where $r = 1,2,\ldots,\ t_1, t_2 = 0,1,2,\ldots$,

Hosking (2007) introduced an analogous expression for TL-moments ,in terms of Jacobi polynomials, given by

$$\lambda_{r+1}^{(t_1,t_2)}=\frac{r!(r+t_1+t_2+1)!}{(r+1)(r+t_1)!(r+t_2)!}\int_0^1 Q(u)u^{t_1}(1-u)^{t_2}P_r^{(t_1,t_2)}(u)du$$
$$=\frac{r!(r+t_1+t_2+1)!}{(r+1)(r+t_1)!(r+t_2)!}E\left[Q(u)u^{t_1}(1-u)^{t_2}P_r^{(t_1,t_2)}(u)\right],\quad r=0,1,... \quad (7),$$

where $P_r^{(t_1,t_2)}(u)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r+t_1}{r-j}\binom{r+t_2}{j}u^j(1-u)^{r-j}$ (8),

is the shifted Jacobi polynomials (see, Abramowitz and Stegun (1972)).

2.1 Sample TL-moments

Let $X_{1:n}\le X_{2:n}\le...\le X_{n:n}$ denote the sample order statistics of a random sample $X_1, X_2,..., X_n$ of size n from the population, the sample TL-moments are defined as an unbiased estimators of the population TL-moments by Elamir and Seheult (2003) given by

$$l_{r+1}^{(t_1,t_2)}=\frac{1}{(r+1)\binom{n}{t_1+t_2+r+1}}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\sum_{i=1}^{n}\binom{i-1}{r+t_1-j}\binom{n-i}{t_2+j}x_{i:n} \quad (9),$$

for $r=0,1,2,...,n$, $t_1=0,1,2,...,n-1$, $t_2=0,1,2,...,n-1$, $t_1+t_2\le n-1$ and $i=1,2,...,n.$

4. Transformations

1. Linear transformation: Let F_x and F_y denote the distribution functions of X and $Y=g(X)$. Similarly, let $Q_X(u)$ and $Q_Y(u)$ denote their quantile functions. If g is an increasing continuous function, then it is well known that
$F_Y(y)=F_X(g^{-1}(y))$

If, further, F_X is strictly increasing continuous distribution, then

i. $Q_X F_X(x)=x$,

ii. $Q_Y(u)=g[Q_X(u)]$

two important consequences of this

1. if $Y=\mu+\sigma X$, then $Q_Y(u)=\mu+\sigma Q_X(u)$.
2. if $Y=\log(X)$, then $Q_Y(u)=\log Q_X(u)$.

in general, if data $X_1,...,X_n$ are assumed to be identically distributed as X, and if the quantile function of X can be transformed to the quantile function of a random variable Y by an increasing continuous transformation g, the the transformed data $g(X_1),...,g(X_n)$ are identically distributed as Y; see, Parzen (1979).

2. Log transformation: This could be suitable for many variables

$$y=\log(x)$$

The advantage of this transformation it could give us good result and we can recover the original estimated $Q_x(u)$ and $f(x)$ as following:

$$x=e^y$$

This imply that

$$Q_X(u)=e^{Q_Y(u)}$$

and the estimated function is:

$$\hat{Q}_X(u)=e^{\hat{Q}_Y(u)}$$

Also we can recover density as following

$$Q'_X(u)=e^{Q_Y(u)}Q'_Y(u)$$

Then

$$f(x)=\frac{f(y)}{e^{Q_Y(u)}}$$

and it is the estimated as:

$$\hat{f}(x)=\frac{\hat{f}(y)}{e^{\hat{Q}_Y(u)}}$$

5. Representation of Density Quantile Function in terms of TL-moments

Hosking (2007),theorem(3), derived the TL-moments as the coefficients in the expansion of the quantile function in the form of a weighted sum of shifted Jacobi polynomials as

$$Q(u)=\sum_{r=0}^{\infty}\frac{(r+1)(2r+t_1+t_2+1)}{r+t_1+t_2+1}\lambda_{r+1}^{(t_1,t_2)}P_r^{(t_1,t_2)}(u),$$

$0\le u\le 1$ (10),

This is convergent in the weighted mean square with weight function $u^{t_1}(1-u)^{t_2}$, i.e.

$$e_s(u)=Q(u)-\sum_{r=0}^{s}c_r P_r^{(t_1,t_2)}(u) \quad (11),$$

the remainder after stopping the sum, after s terms, satisfies

$$\int_0^1 u^{t_1}(1-u)^{t_2}e_s^2(u)du\to 0$$

as $s\to\infty$.

It also represent of density function in terms of TL-moments by the following theorem:

Theorem 1: Let X be a continuous real-valued variable such that its trimmed mean is finite, quantile function $Q(u)=x(F)=F(x)^{-1}$, density function $f(x)$, quantile density function $qu=Q'(u)$, and trimmed L-moments $\lambda_{r+1}^{(t_1,t_2)}, r=0,1,2,......$ then the representation

$$q(u)=Q'(u)=\frac{1}{f(x)}=\sum_{r=1}^{\infty}(r+1)(2r+t_1+t_2+1)\lambda_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u)$$

$, r=1,2,...$ (12),

This is convergent in the weighted mean square with weight function $u^{t_1}(1-u)^{t_2}$.

This imply that

$$f(x)=\frac{1}{\sum_{r=1}^{\infty}(r+1)(2r+t_1+t_2+1)\lambda_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u)} \quad (13).$$

Proof:

Our aim is to minimize $Min.\int_0^1 u^{t_1+1}(1-u)^{t_2+1}e_s^2(u)du$ as following.

$$\int_0^1 u^{t_1+1}(1-u)^{t_2+1}\left[Q'(u)-\sum_{r=1}^{s}c_rP_{r-1}^{(t_1+1,t_2+1)}\right]^2=$$

$$\left[\begin{array}{l}\int_0^1 Q'^2(u)u^{t_1+1}(1-u)^{t_2+1}du-2\int_0^1\sum_{r=1}^{s}c_ru^{t_1+1}(1-u)^{t_2+1}Q'(u)P_{r-1}^{(t_1+1,t_2+1)}(u)du+\\ \int_0^1\sum_{r=1}^{s}\sum_{j=1}^{s}c_rc_ju^{t_1+1}(1-u)^{t_2+1}P_{r-1}^{(t_1+1,t_2+1)}(u)P_{j-1}^{(t_1+1,t_2+1)}(u)du\end{array}\right]$$

$$=\int_0^1 Q'^2(u)u^{t_1+1}(1-u)^{t_2+1}du-2\sum_{r=1}^{s}c_r\frac{(r+1)(r+t_1)!(r+t_2)!}{(r-1)!}\lambda_{r+1}^{(t_1,t_2)}+$$

$$\sum_{r=1}^{s}c_r^2\frac{(r+t_1)!(r+t_2)!}{(r-1)!(2r+t_1+t_2+1)(r+t_1+t_2+1)!} \quad (14).$$

We used the orthogonal property of shifted Jacobi polynomial and where

$$\lambda_{r+1}^{(t_1,t_2)}=\frac{(r-1)!(r+t_1+t_2+1)!}{(r+1)(r+t_1)!(r+t_2)!}E\left[Q'(u)u^{t_1+1}(1-u)^{t_2+1}P_{r-1}^{(t_1+1,t_2+1)}(u)\right]$$

see Hosking (2007). Minimizing this give c_r as

$$c_r=(r+1)(2r+t_1+t_2+1)\lambda_{r+1}^{(t_1,t_2)}$$

With second derivative as

$$2\sum_{r=1}^{s}\frac{(r+t_1)!(r+t_2)!}{(r-1)!(2r+t_1+t_2+1)(r+t_1+t_2+1)!}>0$$

Another Proof:

This proof could be obtained using

$$\frac{d}{du}P_r^{(t_1,t_2)}(u)=(r+t_1+t_2+1)P_{r-1}^{(t_1+1,t_2+1)}(u)$$

Direct using of this in (10) gives

$$Q'(u)=\sum_{r=1}^{\infty}(r+1)(2r+t_1+t_2+1)\lambda_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u)$$

Example:

For the uniform distribution

$f(x)=\frac{1}{b-a}$, for $a<x<b$ the value of

$$\lambda_2=\frac{b-a}{6}$$

If we use (16) for one term

$$f(x)=\frac{1}{6\lambda_2}=\frac{1}{6\frac{b-a}{6}}=\frac{1}{b-a}$$

5 .1 Conditions of density quantile function

Since the quantile function is the left-continuous, we looking for representation of quantile function for given s, t_1 and t_2 in terms of TL-moments

$$Q(u;s,t_1,t_2)=\sum_{r=0}^{s}\frac{(r+1)(2r+t_1+t_2+1)}{r+t_1+t_2+1}\lambda_{r+1}^{(t_1,t_2)}P_r^{(t_1,t_2)}(u)$$

which make $Q(u)$ is monotone increasing function.

Thus our aim is to obtain the quantile function which has first derivative is positive, i.e., the quantile density (sparsity) function $q(u)=Q'(u)$ must be positive

$$q(u)=\sum_{r=1}^{\infty}(r+1)(2r+t_1+t_2+1)\lambda_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u)>0 \quad (15),$$

for all $0<u<1$; see Elamir (2009).

This implies that $f(x)=\frac{1}{Q'(u)}=\frac{1}{q(u)}>0$ (16).

6. Estimation of density quantile function

From a random sample $x_1,x_2,...,x_n$ of size n from the population where its corresponding order statistics is $x_{1:n}<x_{2:n}<....<x_{n:n}$ by replacing the population TL-moments $\lambda_{r+1}^{(t_1,t_2)}$ by it sample version sample TL-moments $l_{r+1}^{(t_1,t_2)}$.

From (12) we can estimate the quantile density function as

$$\hat{q}(u;s,t_1,t_2)=\sum_{r=1}^{s\le n}(r+1)(2r+t_1+t_2+1)l_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u) \quad (17).$$

Where s number of terms, t_1 and t_2 number of trimmed values; see Elamir (2009).

Thus, we can estimate the density as

$$\hat{f}(x)=\frac{1}{\sum_{r=1}^{s}(r+1)(2r+t_1+t_2+1)l_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u)} \quad (18).$$

6.1 Optimal choosing for amount of trimming

Under the two conditions of the density quantile function ,in (15)and(16).The error between the quantile density function and its TL-moments representation can be written as

$$e(u)=q(u)-\sum_{r=1}^{\infty}(r+1)(2r+t_1+t_2+1)\lambda_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u) \quad (19).$$

The optimal values of, t_1 and t_2 can be chosen as the values which have less sum of the absolute error, i.e.,

$$\text{Min}\sum|e(u)| \quad (20)$$

For choosing these values among other values we give two methods.

(a) Method 1 (exact method)

For known distributions we can obtain TL-moments analytically for different values of t_1 and t_2 in the range [(0, 0),…, (4,4)] and evaluate (5.22)for each pair and pick the values which give less sum of the absolute error, in (5.23). But, also, this method is not tractable in most cases and we suggest instead the following approximation method.

(b) Method 2 (approximation method)

For known distribution we may take $u=F$ as one of the plotting positions $E(u)=\frac{i}{n+1}$,

$$Mod(u)=\frac{i-1}{n-1} \quad \text{or} \quad Med(u)\cong\frac{i-0.33}{n+0.38}$$

see Harter (1984). For sufficient n and s (quantile function stopping terms).

For example if we used the $E(u)$,we find

$$Q(u_i)=Q\left(\frac{i}{n+1},\theta\right)$$

Sufficient n and s, fixed data, x, for different values of the parameter(s) θ, compute the error for each pair of the trimming in the range [(0,0),…(4,4)] and pick the pair which has

$$\text{Min}\sum|e_s(u_i)| \quad (21),$$

where

$$e_s(u_i)=q(u_i)-\sum_{r=1}^{s}(r+1)(2r+t_1+t_2+1)l_{r+1}^{(t_1,t_2)}P_{r-1}^{(t_1+1,t_2+1)}(u_i) \quad (22).$$

7. Applications:

In this section, from some known distributions and real data, we investigate the approximation of $f(x)$ by using the expression in (18) that satisfy the two conditions in (15) and (16). For different values of trimming (t_1,t_2) and terms (s), we compute the estimator of absolute weighted mean square error in which given in (22). As the basis of our comparisons, we simulate 50, 100, and 150 observations from normal, uniform and wiebull distributions. In addition, we estimate the density for 155 observations as real data. We take the density which has $\text{Min}\sum|e_s(u_i)|$ and best fitting to distribution.

Note: any figure in this chapter refers to approximation to the density function in (18) in terms of sumulated sample from a particular distribution (or real data) has size n and the trimming values (t_1,t_2). Letter (a) refers to the density in (18) as function of $\hat{u} = p$. Letter (b) refers to the density in (18) as function of $\hat{Q}(u)$. Letter (c) refers to the density in (18) as function of $\hat{Q}(u)$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). Letter (d) refers to the density in (18) as function of $x_{1:n}$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

7.1 Normal distribution

Normal distributions are extremely important in statistics and are often used in the natural and social sciences for real-valued random variables whose distributions are not known.

If X is distributed according to the normal distribution, then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0,$$

$Q(u)$ has no explicit analytical form.

The normal distribution is immensely useful because of the central limit theorem, which states that, under mild conditions, the mean of many random variables independently drawn from the same distribution is distributed approximately normally, irrespective of the form of the original distribution.

Moreover, many results can be derived analytically in explicit form when the relevant variables are normally distributed.

We simulate 50, 100, and 150 observations from normal distribution with $\mu = 0$ and $\sigma = 1$. The data is given in table 1.

Table 1: simulated data from normal distribution with $\mu = 0$ and $\sigma = 1$, n=50,100,150.

				n=50				
0.3549	-0.5633	-0.5868	0.2196	1.6200	0.8166	0.0372	-1.5988	0.3521
-0.5070	1.3854	1.9735	0.4293	0.3488	0.1726	-0.3505	1.0655	-1.1045
-0.9097	-1.4698	0.4051	0.3262	0.2485	0.5195	-0.2100	-0.4667	-1.5118
1.3579	-0.3441	-1.6407	0.4410	-1.4366	-0.0118	-1.3796	1.4222	-1.0835
0.6253	0.0211	-0.8047	-1.5906	0.6408	-2.2141	1.8542	0.0058	-0.0878
-0.5267	-0.5568	0.3613	-0.4770	0.6527				
				n=100				
0.8420	1.0430	-0.5103	1.6114	0.0032	-0.2481	0.0786	0.6924	-0.7444
-0.6254	-0.8520	-0.1113	2.6477	-0.2475	-0.6439	1.1295	0.8143	-0.2690
0.0601	-1.0971	0.2650	0.1584	-0.3494	0.4629	-0.7668	0.1794	-0.22194
1.0209	0.7938	-0.6266	0.1516	-0.4708	-1.1614	-0.8851	0.3673	-0.2190
0.6292	-1.4979	0.3945	1.0745	0.5428	-0.0203	-0.7404	1.2532	-1.4061
0.8817	-0.1522	1.7715	0.4583	-1.1190	1.7167	0.7836	-0.0906	-0.3385
-0.9317	0.4524	-0.0986	-0.6036	0.3736	0.1235	-0.3030	1.1967	-1.0428
-0.0586	0.1399	1.1942	-2.4827	0.15588	0.4243	-2.0982	-2.0507	-1.1512
0.2379	-1.4437	-0.4069	0.3120	-2.6962	0.6798	1.08147	-0.7856	0.8030
-1.1845	0.3028	0.2052	0.7050	-1.1230	1.2701	2.3199	0.0725	-0.9135
0.0125	-0.0596	0.0593	0.2881	-0.5718	1.3987	-0.6956	-2.4769	-1.5394
-1.7465								
				n=150				
0.8420	1.0430	-0.5101	1.6114	0.0032	-0.2481	0.0786	0.6924	-0.6254
-0.8520	-0.1113	2.6477	-0.2475	-0.6439	1.1295	0.8143	-0.2690	-1.0971
0.2651	0.1584	-0.3494	0.4629	-0.7668	0.1794	-0.2219	1.0209	-0.6266
0.1516	-0.4708	-1.1614	-0.8851	0.3673	-0.2190	0.6292	-1.4979	1.0745
0.5427	-0.0203	-0.7404	1.2532	-1.4061	0.8818	-0.1522	1.7715	-1.1190
1.7167	0.7836	-0.0906	-0.3385	-0.9317	0.4524	-0.0985	-0.6036	0.1235
-0.3030	1.1966	-1.0428	-0.0586	0.1399	1.1942	-2.4827	0.1559	-2.0982

-2.0508	-1.1512	0.23789	-1.4437	-0.4069	0.3120	-2.6962	0.6798	-0.7856
0.8030	-1.1845	0.3028	0.2052	0.7050	-1.1230	1.2701	2.3199	0.0725
0.0125	-0.0596	0.0593	0.2881	-0.5718	1.3988	-0.6956	-2.4769	-1.5394
-0.6981	-0.2524	1.0511	0.0106	-0.2639	0.4627	-0.9513	-0.5595	-0.2156
-0.7253	1.0612	-2.2149	0.7282	-0.9523	-0.0207	0.6105	-0.3795	-1.1546
-1.6338	0.5560	-1.3143	-1.4933	0.8479	0.2885	-0.0256	-0.0125	0.0354
-0.6143	0.7862	0.3088	-0.7100	1.0977	0.6628	0.0158	0.9268	0.0012
-1.5720	0.7628	-0.2212	-1.1199	-1.3154	0.0818	-0.6873	-2.1367	-1.5553
-0.7444	0.0601	0.7938	0.3943	1.0814	0.3736	0.4243	0.4582	-0.2989
-0.9135	0.9101	-0.2861	-2.0011	-1.7465	-0.5399			

The values of $\sum |e_s(u_i)|$ for different trimming and terms for three sample size of normal $(0,1)$ distribution are given in Table2. We concluded, for three sample size, the good approximation of density which has less error given at trimming $(t_1 = 1, t_2 = 1)$ and the terms $s = 4$. This approximation is shown, for the three simulated sample sizes, in Figure 1,2 and 3, respectively.

Table 2: values of $\sum |e_s(u_i)|$ for different choices of trimming (t_1, t_2) and terms s from normal distribution with $\mu = 0$ and $\sigma = 1$ using n=50,100,150.

n=50

(t_1, t_2)	(0,0)	(1,0)	**(1,1)**
s=4	0.1250	0.1608	**0.00597***
s=6	0.2568	0.2244	**0.0378**
s=7	0.2942	0.2846	**0.0429**

n=100

(t_1, t_2)	(0,0)	(0,1)	(1,0)	**(1,1)**
s=4	0.0728	0.1029	0.1756	**0.0250***
s=6	0.2604	0.2811	0.0203	**0.0285**
s=8	0.6442	0.3965	0.3051	**0.0572**
s=10	0.8163	0.4145	0.4518	**0.0628**

n=150

(t_1, t_2)	(0,0)	(0,1)	(1,0)	**(1,1)**	(2,2)	(2,1)	(1,2)	(3,1)
s=4	0.3866	0.0093	0.3959	**0.0018***	0.0030	0.00431	0.0063	0.0074
s=6	0.1789	0.0891	0.2624	**0.0033**	0.0031	0.00430	0.0081	0.0073

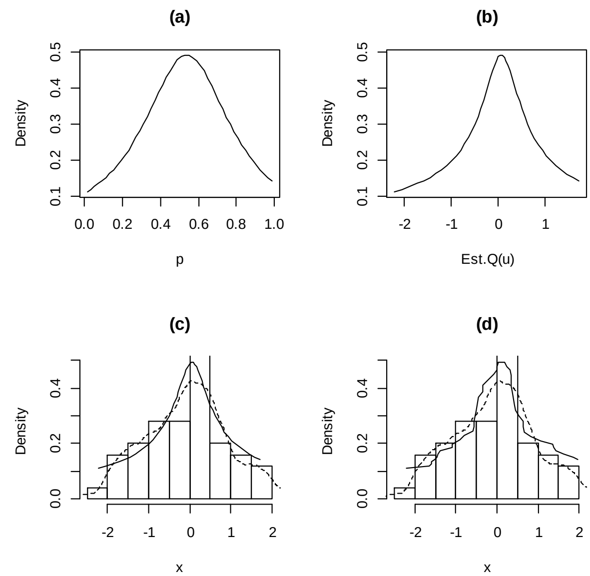

Figure 1: Approximation to the density function of normal distribution ($\mu = 0, \sigma = 1$), using terms ($s = 4$), trimming $(t_1 = 1, t_2 = 1)$ and sample size n=50. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}\ (x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

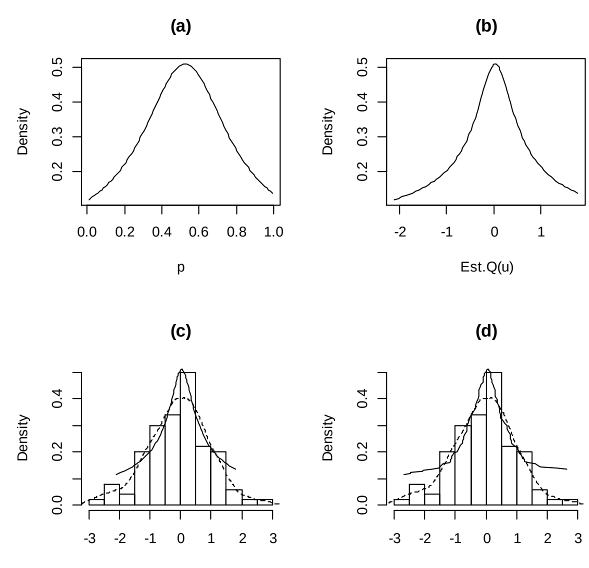

Figure 2: Approximation to the density function of normal distribution ($\mu = 0, \sigma = 1$), using terms ($s = 4$), trimming $(t_1 = 1, t_2 = 1)$ and sample size n=100. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}\ (x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

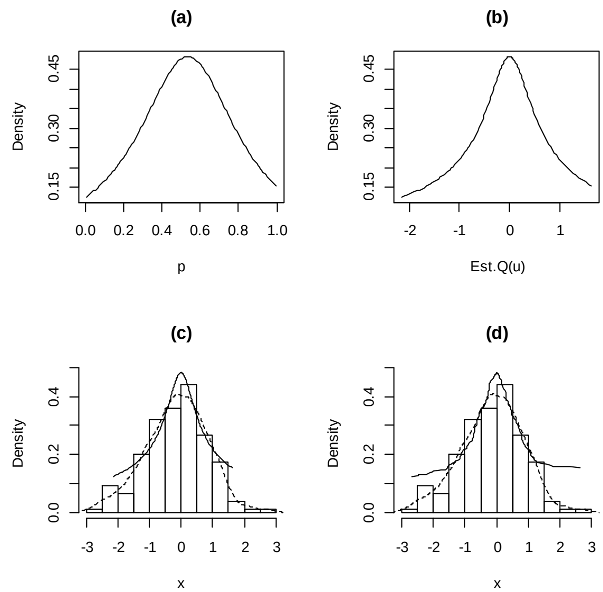

Figure 3: Approximation to the density function of normal distribution ($\mu = 0$, $\sigma = 1$), using terms ($s = 4$), trimming $(t_1 = 1, t_2 = 1)$ and sample size n=150. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}(x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

7.2 Uniform distribution

If X is distributed according to the uniform distribution, then

$$f(x) = \frac{1}{\beta - \alpha}, \quad \alpha < x < \beta \quad ,$$

and

$$Q(u) = \alpha + (\beta - \alpha)u$$

We assume that $\alpha = 0$ and $\beta = 1$.

We simulate 50, 100, and 150 observations from uniform distribution with $\alpha = 0$ and $\beta = 1$. The data is given in table 3.

Table 3: simulated data from uniform distribution with $\alpha = 0$ and $\beta = 1$, n=50,100,150.

				n=50				
0.7397	0.0394	0.3372	0.6318	0.3817	0.1410	0.1563	0.3967	0.7981
0.7421	0.0705	0.5566	0.2107	0.7647	0.5649	0.7570	0.1638	0.5093
0.3833	0.1531	0.7907	0.7572	0.7485	0.3810	0.8582	0.2306	0.0585
0.9873	0.4078	0.6634	0.4820	0.7281	0.1120	0.3330	0.8385	0.8480
0.4039	0.0310	0.0616	0.3835	0.7647	0.3322	0.6874	0.0764	0.9899
0.9901	0.9317	0.3298	0.9391	0.8619				
				n=100				
0.3081	0.6887	0.7987	0.9906	0.0882	0.7505	0.8300	0.8280	0.8100
0.1710	0.2955	0.8360	0.2051	0.2202	0.4448	0.9585	0.0204	0.1329
0.2514	0.4719	0.1376	0.7555	0.3618	0.6468	0.5548	0.1118	0.1112
0.4867	0.7343	0.3828	0.1440	0.8923	0.1735	0.5446	0.5195	0.1820

0.9953	0.5814	0.2213	0.1156	0.4759	0.4795	0.6595	0.5602	0.6547
0.1077	0.6686	0.3914	0.7122	0.0319	0.2203	0.2619	0.5639	0.7655
0.3102	0.5632	0.6574	0.5318	0.7696	0.2214	0.1319	0.0697	0.0101
0.4885	0.7622	0.5891	0.7929	0.5109	0.0607	0.3922	0.9101	0.6685
0.2235	0.0378	0.9277	0.7136	0.0662	0.6015	0.0202	0.0709	0.4917
0.8369	0.5729	0.5774	0.4494	0.1115	0.5186	0.8570	0.9777	0.5939
0.3500	0.7007	0.8939	0.1180	0.0328	0.8057	0.1744	0.5940	0.3406
0.8530								
				n=150				
0.3379	0.9636	0.3192	0.9392	0.0313	0.3302	0.2900	0.4562	0.4075
0.6874	0.1743	0.3241	0.8260	0.6385	0.4492	0.6346	0.2385	0.0449
0.6799	0.8376	0.3884	0.8012	0.3001	0.5275	0.7286	0.6477	0.4416
0.3352	0.5384	0.8783	0.1239	0.7785	0.2795	0.8970	0.4885	0.2446
0.1217	0.9962	0.7012	0.1623	0.1536	0.4003	0.6248	0.9394	0.8825
0.2771	0.9918	0.3180	0.8247	0.8171	0.3336	0.1508	0.5373	0.3563
0.1484	0.7780	0.8279	0.1389	0.7111	0.1308	0.2975	0.7603	0.7935
0.1302	0.3653	0.3046	0.9764	0.7510	0.4062	0.5264	0.7627	0.0748
0.1897	0.0223	0.9750	0.6474	0.6806	0.2391	0.2322	0.7982	0.1176
0.9165	0.7582	0.1944	0.3996	0.7691	0.1331	0.9181	0.0849	0.9607
0.1417	0.6917	0.7158	0.5219	0.4781	0.9327	0.9042	0.0242	0.6676
0.6879	0.3745	0.3314	0.2097	0.8487	0.9792	0.3264	0.1892	0.7986
0.7917	0.8214	0.3180	0.0765	0.6643	0.2085	0.5440	0.7711	0.3283
0.4831	0.4753	0.1624	0.9041	0.9491	0.7046	0.8465	0.8522	0.3022
0.6651	0.0331	0.6820	0.1158	0.8308	0.2663	0.8567	0.7668	0.9065
0.2131	0.3742	0.5921	0.5873	0.4770	0.2761	0.6104	0.9198	0.3301
0.1771	0.2834	0.6553	0.0451	0.8218	0.6173			

We note that all the values of $\sum |e_s(u_i)|$ for different trimming and terms for three sample sizes of uniform $(0,1)$ distribution, given in Table 4, are very close to zero. By comparing the densities we concluded, for the three sample sizes, the good approximation of density at terms $s = 4$. For sample of size n=50, it's given at trimming $(t_1 = 1, t_2 = 0)$. While with increasing the sample of size, it's given at trimming $(t_1 = 0, t_2 = 0)$.This approximation is shown, for the three simulated sample sizes, in Figure 4, 5 and 6, respectively. Also, we note that the new proposed estimator give an approximation best of the kernel estimator.

Table 4: values of $\sum\left|e_S(u_i)\right|$ for different choices of trimming (t_1,t_2) and terms S from uniform distribution with $\alpha=0$ and $\beta=1$ using n=50,100,150.

n=50			
(t_1,t_2)	(0,0)	**(1,0)**	(0,1)
r=2	0.0216	**0.0075***	0.0140
r=3	0.0212	**0.0067**	0.0140
r=4	0.0193	**0.0066**	0.0122

n=100				
(b,a)	**(0,0)**	(0,1)	(1,0)	(1,1)
r=2	**0.0086***	0.0061	0.00245	0.0021
r=3	0.0086	0.0061	0.00242	0.0020
r=4	0.0083	0.0043	0.00190	0.00189

n=150				
(t_1,t_2)	**(0,0)**	(1,1)	(1,0)	(0,1)
r=2	**0.0224***	0.0053	0.0096	0.0129
r=3	0.0224	0.0053	0.0091	0.0127
r=4	0.0204	0.0044	0.0082	0.0122
r=6	0.0136	0.0030	0.0050	0.0086

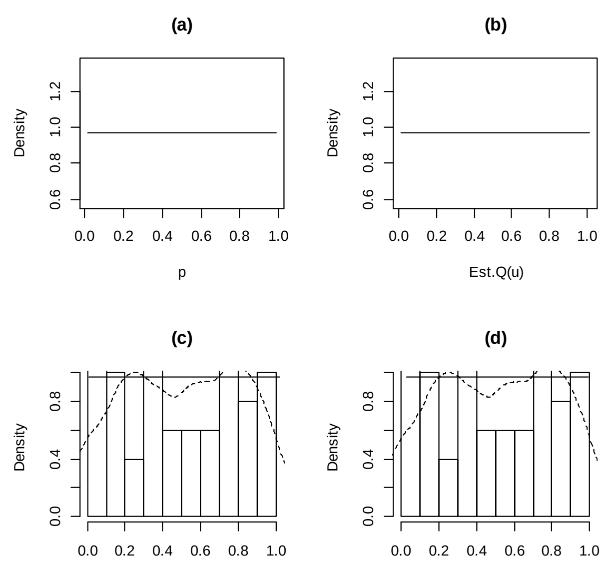

Figure 4: Approximation to the density function of uniform distribution with $\alpha=0$ and $\beta=1$, using terms ($s=2$), trimming $(t_1=1,t_2=0)$ and sample size n=50. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}(x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

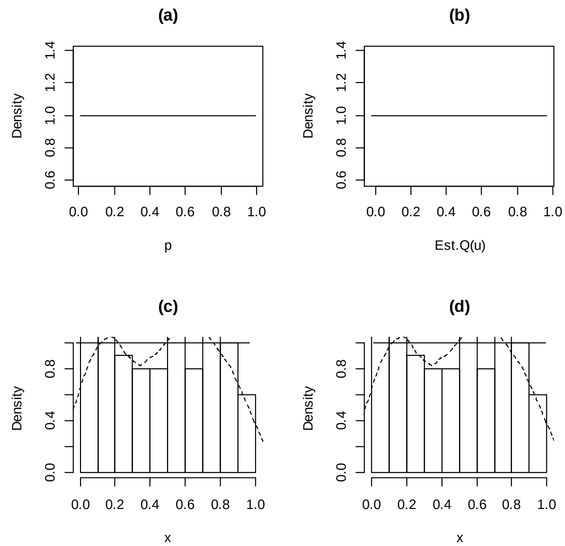

Figure 5: Approximation to the density function of uniform distribution with $\alpha = 0$ and $\beta = 1$, using terms ($s = 2$), trimming $(t_1 = 0, t_2 = 0)$ and sample size n=100. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}(x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

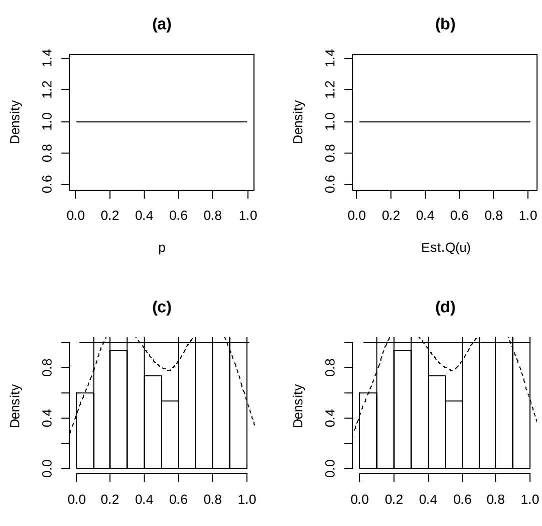

Figure 5: Approximation to the density function of uniform distribution with $\alpha = 0$ and $\beta = 1$, using terms ($s = 2$), trimming $(t_1 = 0, t_2 = 0)$ and sample size n=150. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}(x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

7.3 Weibull distribution

The probability density function is

$$f(x)=\begin{cases}\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k} & x\geq 0,\\ 0 & x<0\end{cases}$$

where $\lambda>0, k>0$ are the scale and shape parameters.

Also, the quantile function of a Weibull random variable is

$$Q(u)=\lambda(-\ln(1-u))^{1/k},$$

We simulate 50, 100, and 150 observations from Weibull distribution with scale and shape parameters ($\lambda=1, k=1.5$). The data is given in table 5.

Table 5: simulated data from Weibull distribution with scale and shape parameters ($\lambda=1, k=1.5$), n=50,100,150.

n=50											
1.424	0.327	1.364	0.482	1.139	1.092	1.558	0.681	1.164	1.123	1.275	0.139
0.339	1.271	0.588	0.027	0.053	0.760	1.383	0.246	0.930	1.034	1.252	0.929
1.007	0.403	1.771	0.538	0.781	1.214	0.213	0.921	0.699	0.752	1.777	0.909
2.203	0.814	0.228	0.515	0.753	0.501	0.142	0.553	0.457	1.271	0.235	0.495
1.747	1.261										
n=100											
1.342	0.801	1.561	0.834	0.468	0.471	1.165	0.167	0.213	0.688	0.947	0.928
1.429	1.526	1.448	0.427	1.169	0.583	0.997	0.110	1.640	1.035	1.381	0.377
1.046	0.670	1.225	0.941	1.506	0.224	0.796	0.838	0.215	0.605	1.604	0.565
0.668	1.451	1.095	0.254	1.642	1.850	0.082	0.547	0.590	1.081	0.095	0.233
0.238	0.443	0.205	0.264	0.779	0.179	0.833	0.342	0.764	0.203	0.579	0.393
2.905	0.454	2.241	1.440	0.198	0.702	0.825	0.653	1.938	1.539	1.743	0.135
1.547	0.290	0.398	1.256	0.908	0.631	0.044	0.180	0.398	1.321	0.992	1.074
1.418	0.274	1.431	1.357	0.354	1.611	1.847	0.144	0.215	0.365	0.666	0.786
2.602	0.684	0.774	3.272								
n=150											
1.261	0.345	0.407	0.511	0.779	0.266	1.236	0.512	0.414	0.286	0.781	0.220
2.006	0.550	1.702	1.524	0.917	1.214	0.198	0.333	1.015	0.586	0.312	0.315
0.742	0.875	0.228	0.141	0.481	0.393	0.625	0.333	1.146	0.513	0.578	1.245
0.625	0.626	1.313	1.446	0.976	0.333	1.180	1.081	1.425	0.442	0.289	0.858
0.600	0.658	1.588	0.491	0.363	0.664	1.184	1.032	0.187	0.581	0.553	0.664
0.668	1.098	1.077	2.659	0.424	1.007	0.606	0.832	0.567	0.444	0.460	0.626
0.055	1.287	1.200	1.683	0.970	2.290	0.135	0.295	0.447	0.928	0.140	0.836
1.539	0.844	0.239	0.377	0.684	1.016	0.836	0.364	1.384	1.258	0.826	1.849
0.517	0.204	0.179	0.419	1.803	0.755	0.633	0.567	2.704	0.548	0.435	0.336
0.469	1.315	0.702	1.554	0.819	1.600	0.499	0.221	0.805	1.127	0.968	0.631
1.856	0.817	1.843	2.162	0.524	2.053	0.909	0.668	0.161	1.656	1.629	0.282
1.286	0.634	0.427	0.380	1.972	0.944	0.105	0.385	0.036	0.759	0.461	0.480
0.841	1.123	0.305	0.834	0.567	1.666						

Table 6: values of $\sum|e_s(u_i)|$ for different choices of trimming (t_1,t_2) and terms S from Weibull distribution with scale and shape parameters ($\lambda=1, k=1.5$) using n=50,100,150.

n=50

(t_1,t_2)	(0,0)	(1,0)	**(0,1)**
s=4	0.084	0.087	**0.015***
s=6	0.024	0.020	**0.019**

n=100

(t_1,t_2)	(0,0)	(0,1)	(1,1)	**(2,1)**
s=4	0.413	0.022	0.020	**0.018***
s=6	0.106	0.021	0.019	**0.017**

n=150

(t_1,t_2)	(0,0)	(0,1)	(1,1)	**(1,2)**	(0,2)	(0,3)	(0,4)
s=4	0.0729	0.0137	0.0041	**0.0022***	0.0095	0.0074	0.0064
s=6	0.0110	0.0156	0.0053	**0.0023**	0.0103	0.0081	0.0071
s=8	0.0436	0.0167	0.0058	**0.0023**	0.0109	0.0087	0.0076
s=10	0.1006	0.0174	0.0061	**0.0024**	0.0113	0.0090	0.0080

The values of $\sum|e_s(u_i)|$ |for different trimming and terms for three sample size of Weibull distribution with scale and shape parameters ($\lambda=1, k=1.5$) are given in Table 6. We concluded the good approximation of the density which has less error given at trimming $(t_1=0,t_2=1)$, $(t_1=2,t_2=1)$ and $(t_1=1,t_2=2)$ for three sample sizes respectively and the terms $s=4$. This approximation of the density is shown, for the three simulated sample sizes, in Figure 7, 8 and 9, respectively.

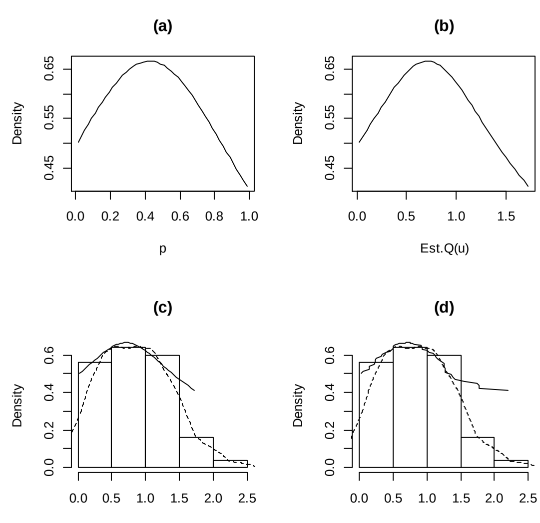

Figure 7: Approximation to the density function of wiebull distribution ($\lambda=1, k=1.5$), using terms ($s=4$), trimming $(t_1=0,t_2=1)$ and sample size n=50. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}\ (x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

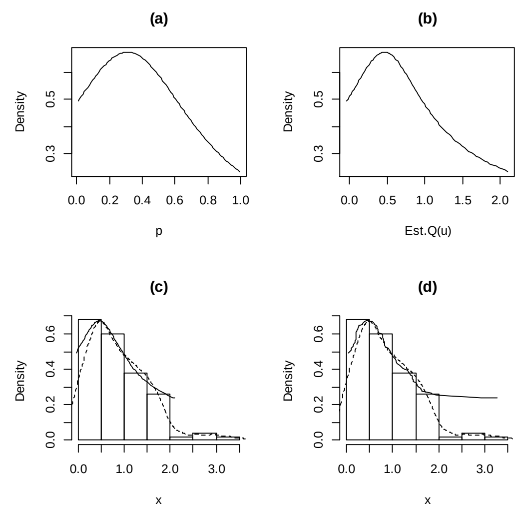

Figure 8: Approximation to the density function of wiebull distribution ($\lambda = 1, k = 1.5$), using terms ($s = 4$), trimming $(t_1 = 2, t_2 = 1)$ and sample size n=50. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}(x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

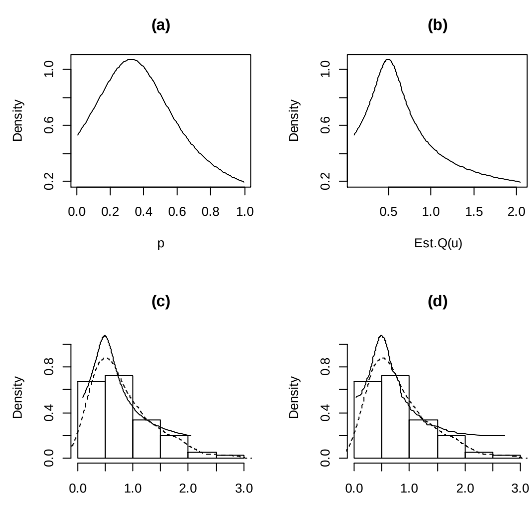

Figure 9: Approximation to the density function of wiebull distribution ($\lambda = 1, k = 1.5$), using terms ($s = 4$), trimming $(t_1 = 1, t_2 = 2)$ and sample size n=150. (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}(x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

7.4 Data

The following data represent the ages for 155 patients of Breast Tumors taken from (June-November 2014), whose entered in (), are listed in Table 7

Table 7: data of the ages for 155 patients of Breast Tumors taken from (June-November 2014), whose entered in (), in Egypt.

46	32	50	46	44	42	69	31	25	29	40	42	24	17	35
48	49	50	60	26	36	56	65	48	66	44	45	30	28	40
40	50	41	39	36	63	40	42	45	31	48	36	18	24	35
30	40	48	50	60	52	47	50	49	38	30	52	52	12	48
50	45	50	50	50	53	55	38	40	42	42	32	40	50	58
48	32	45	42	36	30	28	38	54	90	80	60	45	40	50
50	40	50	50	50	60	39	34	28	18	60	50	20	40	50
38	38	42	50	40	36	38	38	50	50	31	59	40	42	38
40	38	50	50	50	40	65	38	40	38	58	35	60	90	48
58	45	35	38	32	35	38	34	43	40	35	54	60	33	35
36	43	40	45	56										

The values of $\sum\left|e_s(u_i)\right|$ |for different trimming and terms for data of the age of 155 patients of Breast Tumors are given in Table 8. The approximation of density which has less error given at trimming $(t_1 = 1, t_2 = 2)$. By comparing the behavior of this density at terms $s = 4,6 \text{ and } 7$, showed in Figures 10, 11 and 12, respectively, with the behavior of the approximation density of normal distribution at the same terms. We concluded that normal distribution is a good model for the above data. Thus, the good approximation of the density for the above data of the age for 155 patients of Breast Tumors is given in Figure 10, at terms $s = 4$ and trimming $(t_1 = 1, t_2 = 2)$.

Table 8: Values of $\sum\left|e_s(u_i)\right|$ for different choices of trimming (t_1, t_2) and terms s from data of the age for 155 patients of Breast Tumors.

(t_1, t_2)	(0,0)	(0,1)	(1,1)	**(1,2)**	(2,1)
s=4	642.17	174.84	69.40	**29.86***	39.54
s=6	472.54	138.32	62.39	**27.67**	34.10
s=7	441.91	138.32	60.50	**26.83**	33.65

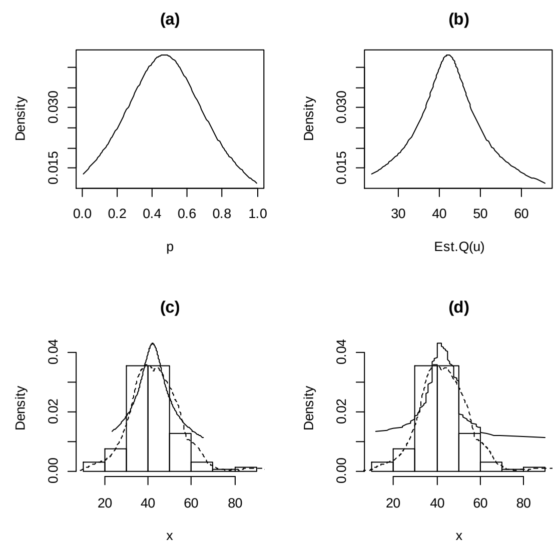

Figure 10: Approximation to the density quantile function of Breast Tumors data, sample size n=155, trimming $(t_1 = 1, t_2 = 2)$ and using four terms ($s = 4$).(a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}\ (x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

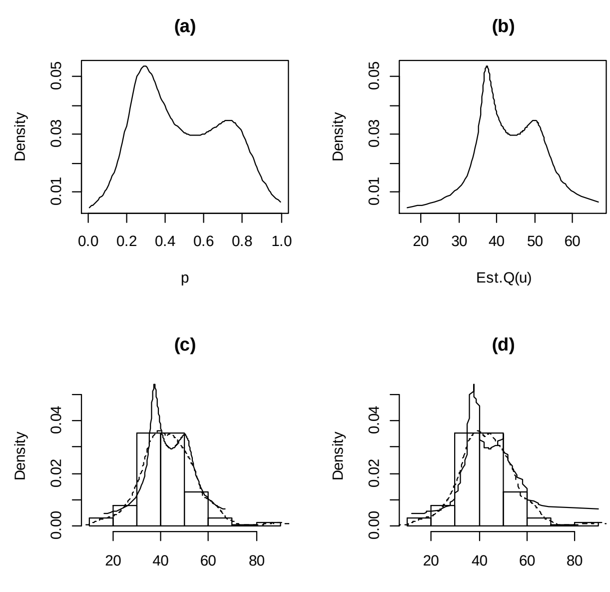

Figure 11: Approximation to the density quantile function of Breast Tumors data, sample size n=155, trimming $(t_1 = 1, t_2 = 2)$ and using six terms ($s = 6$).(a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}\ (x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

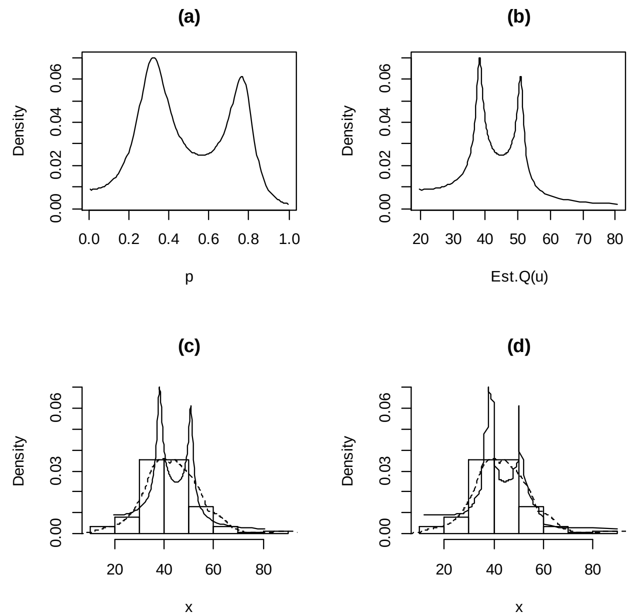

Figure 10: Approximation to the density quantile function of Breast Tumors data, sample size n=155, trimming $(t_1 = 1, t_2 = 2)$ and using seven terms ($s = 7$). (a) The density $\hat{f}_u(u)$. (b) The density quantile $\hat{f}(Q(u))$. (c) The density quantile $\hat{f}(Q(u))$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line). (d) The density $\hat{f}\ (x_{1:n})$ (solid line), histogram and kernel density with Epanechnikov kernel and Silverman's rule of thumb bandwidth (dotted line).

8 Conclusion

We studied nonparametric technique based on TL-moments and orthogonal Jacobi polynomial as an approximation to population density function. This technique has the ability to capture more information about arbitrary distribution that generated the data. The technique is based on minimizing the weighted mean square error between the population quantile density function and its TL-moments representation. Unlike parametric counterpart, no prior assumption of the underlying distribution is required.

We gave some criteria for choosing number of terms and trim values to help in obtaining a good fit for the data. Also we illustrated the benefits of the proposed method using symmetric and asymmetric distributions.

References

Abramowitz, M. and Stegun, I. A., 1972. *Handbook of Mathematical Functions*. Washington, D.C.: National Bureau of Standards.

Bera, A. and Bilias, Y., 2002. The MM, ME, ML, EL, EF and GMM approaches to estimation: A synthesis. *Journal of Econometrics*, **107**, 51-86.

Box, G. E. P. and Cox, D. R., 1964. An analysis of transformations. Journal of the Royal Statistical Society ,B, 26, 211-252.

Cheng, C., 2002. Almost-sure uniform error bounds of general smooth estimators of quantile density functions. Statistics & Probability Letters, 59, 183–194.

Cheng, C. and Parzen, E., 1997. Unified estimators of smooth quantile and quantile density functions. J. Statist. Plann. Inference 59, 291–307.

David, H. A., 2003. Order Statistics. 2nd ed., Wiley, New York.

Elamir, E. A. H. and Seheult, A. H., 2003. Trimmed L-moments. Computational Statistics Data Analysis 43, 299-314.

Elamir, E. A. H. and Seheult, A. H., 2004. Exact covariance of sample L-moments. J. statist. Plann. Inference 124, 337-359.

Elamir, E. A. H., 2009. Nonparametric Estimation for Quantile and Sparsity Functions via Trimmed L-moments. Essays on Mathematics and Statistics, Edited by Vladimir Akis, published by ATINER, 195-210.

Elamir, E. A. H., 2010. Optimal Choices for Trimming in Trimmed L-moment Method. Applied Mathematical Sciences, 4, 2881 – 2890.

Embrechts, P., Kluppelberg, C. and Mikosch, T., 1997. Modeling Extremal Events. Berlin: Springer-Verlag.

Gilchrist, W. G., 2000. Statistical modelling with quantile functions. Chapman and Hall, New York.

Harter, H. L., 1984. Another look at plotting positions. Communication in Statistics-Theory and Methods, 13, 1613-1633.

Hosking, J. R. M., 1990. L-moments: analysis and estimation of distributions using linear combinations of statistics. J. Roy. Statist. Soc. B 52,105-124.

Hosking, J.R.M., 2007. Some theory and practical uses of trimmed L-moments. J. statist. Plann. Inference. 137, 3024-3039.

Hosking, J. R. M. and Wallis, J. R., 1993. Some statistics useful in regional frequency analysis. Water Resour. Res., 29, 271 –281.

Hosking, J. R. M. and Wallis, J. R., 1997. Regional frequency analysis: An approach based on L-moments. University press, Cambridge.

Jones, M. C., 1992. Estimating densities, quantiles, quantiles densities and density quantiles. Annals of the Institute of Statistical Mathematics 44, 721-727.

Karian, A. and Dudewicz, E. J., 2000. Fitting statistical distributions . CRC Press.

Leonowicz, Z., Karvanen, J. and Shishkin, S. L. (2005). Trimmed estimators for robust averaging of event-related potentials. Journal of Neuroscience Methods, 142, 16-26.

Parzen, E., 1979. Nonparametric statistical data modeling. Journal of Statistical Association, 74, 105-121.

Serfling, R. J., 1980. Approximation Theorems of Mathematical Statistics. Wiley, New York.

Silverman, B. W., 1986. Density Estimation for Statistics and Data Analysis. Chapman and Hall, London.

Willinger, W., Paxson, V., and Taqqu, M. S., 1998. Self-similarity and heavy tails; Structural modeling of network traffic. In A Practical Guide to Heavy Tails, eds. Adler, R., Feldman, R., and Taqqu, M.S., pp. 27-53. Boston: Birkhauser.